

Some relations between $t(a, b, c, d; n)$ and $N(a, b, c, d; n)$

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Abstract

Let \mathbb{Z} and \mathbb{N} be the set of integers and the set of positive integers, respectively. For $a, b, c, d, n \in \mathbb{N}$ let $N(a, b, c, d; n)$ be the number of representations of n by $ax^2 + by^2 + cz^2 + dw^2$, and let $t(a, b, c, d; n)$ be the number of representations of n by $ax(x-1)/2 + by(y-1)/2 + cz(z-1)/2 + dw(w-1)/2$ ($x, y, z, w \in \mathbb{Z}$). In this paper we reveal some connections between $t(a, b, c, d; n)$ and $N(a, b, c, d; n)$.

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1. Introduction

Let \mathbb{Z} and \mathbb{N} be the set of integers and the set of positive integers, respectively. Let $\mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{N}^4 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. For $n \in \mathbb{N}$ let

$$\sigma(n) = \sum_{d|n, d \in \mathbb{N}} d.$$

For convenience we define $\sigma(n) = 0$ for $n \notin \mathbb{N}$. For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$ set

$$N(a, b, c, d; n) = |\{(x, y, z, w) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dw^2\}|$$

and

$$t(a, b, c, d; n) = \left| \left\{ (x, y, z, w) \in \mathbb{Z}^4 \mid n = a \frac{x(x-1)}{2} + b \frac{y(y-1)}{2} + c \frac{z(z-1)}{2} + d \frac{w(w-1)}{2} \right\} \right|.$$

The numbers $\frac{x(x-1)}{2}$ ($x \in \mathbb{Z}$) are called triangular numbers.

In 1828 Jacobi showed that

$$N(1, 1, 1, 1; n) = 8 \sum_{d|n, 4 \nmid d} d.$$

In 1847 Eisenstein (see [D]) gave formulas for $N(1, 1, 1, 3; n)$ and $N(1, 1, 1, 5; n)$. From 1859 to 1866 Liouville made about 90 conjectures on $N(a, b, c, d; n)$ in a series of papers.

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Most conjectures of Liouville have been proved. See [A1, A2, AALW1-AALW5], Cooper's survey paper [C], Dickson's historical comments [D] and Williams' book [W3].

Let

$$t'(a, b, c, d; n) = \left| \left\{ (x, y, z, w) \in \mathbb{N}^4 \mid n = a \frac{x(x-1)}{2} + b \frac{y(y-1)}{2} + c \frac{z(z-1)}{2} + d \frac{w(w-1)}{2} \right\} \right|.$$

As $\frac{x(x-1)}{2} = \frac{(-x+1)(-x)}{2}$ we have

$$(1.1) \quad t(a, b, c, d; n) = 16t'(a, b, c, d; n).$$

In [L] Legendre stated that

$$(1.2) \quad t'(1, 1, 1, 1; n) = \sigma(2n + 1).$$

In 2003, Williams [W1] showed that

$$(1.3) \quad t'(1, 1, 2, 2; n) = \frac{1}{4} \sum_{d|4n+3} (d - (-1)^{\frac{d-1}{2}}).$$

For $a, b, c, d \in \mathbb{N}$ with $5 \leq a + b + c + d \leq 8$ let

$$C(a, b, c, d) = 16 + 4i_1(i_1 - 1)i_2 + 8i_1i_3,$$

where i_j is the number of elements in $\{a, b, c, d\}$ which are equal to j . When $a + b + c + d \in \{5, 6, 7\}$, in 2005 Adiga, Cooper and Han [ACH] showed that

$$(1.4) \quad C(a, b, c, d)t'(a, b, c, d; n) = N(a, b, c, d; 8n + a + b + c + d).$$

When $a + b + c + d = 8$, in 2008 Baruah, Cooper and Han [BCH] proved that

$$(1.5) \quad C(a, b, c, d)t'(a, b, c, d; n) = N(a, b, c, d; 8n + 8) - N(a, b, c, d; 2n + 2).$$

In 2009, Cooper [C] determined $t'(a, b, c, d; n)$ for $(a, b, c, d) = (1, 1, 1, 3), (1, 3, 3, 3), (1, 2, 2, 3), (1, 3, 6, 6), (1, 3, 4, 4), (1, 1, 2, 6)$ and $(1, 3, 12, 12)$.

In [WS], Wang and Sun obtained explicit formulas for $t(a, b, c, d; n)$ in the cases $(a, b, c, d) = (1, 2, 2, 4), (1, 2, 4, 4), (1, 1, 4, 4), (1, 4, 4, 4), (1, 3, 3, 9), (1, 1, 9, 9), (1, 9, 9, 9), (1, 1, 1, 9), (1, 3, 9, 9)$ and $(1, 1, 3, 9)$.

Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \quad (|q| < 1).$$

It is evident that for $|q| < 1$,

$$(1.6) \quad \sum_{n=0}^{\infty} N(a, b, c, d; n) q^n = \varphi(q^a) \varphi(q^b) \varphi(q^c) \varphi(q^d),$$

$$(1.7) \quad \sum_{n=0}^{\infty} t'(a, b, c, d; n) q^n = \psi(q^a) \psi(q^b) \psi(q^c) \psi(q^d).$$

From [BCH, Lemma 4.1] or [Be] we know that for $|q| < 1$,

$$(1.8) \quad \psi(q)^2 = \varphi(q)\psi(q^2).$$

$$(1.9) \quad \varphi(q) = \varphi(q^4) + 2q\psi(q^8),$$

$$(1.10) \quad \varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2,$$

$$(1.11) \quad \psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}).$$

By (1.9), for $k \in \mathbb{N}$,

$$(1.12) \quad \varphi(q^k) = \varphi(q^{4k}) + 2q^k\psi(q^{8k}) = \varphi(q^{16k}) + 2q^{4k}\psi(q^{32k}) + 2q^k\psi(q^{8k}).$$

In this paper, using (1.8)-(1.12) we reveal some connections between $t(a, b, c, d; n)$ and $N(a, b, c, d; n)$. Suppose $k, m \in \{0, 1, 2, \dots\}$, $a, n \in \mathbb{N}$ and $2 \nmid a$. We show that

$$t(a, b, c, d; n) = \frac{2}{3} (N(a, b, c, d; 8n + a + b + c + d) - N(a, b, c, d; 2n + (a + b + c + d)/4))$$

for $(a, b, c, d) = (a, a, 2a, 8m + 4)$ and $(a, 3a, 4k + 2, 4m + 2)$ with $k \equiv m \pmod{2}$. For $2 \nmid ak$ we show that

$$t(a, 3a, k, k; m) = \frac{2}{3} N(a, 3a, 2k, 2k; 8m + 4a + 2k).$$

For $n \equiv k + \frac{a-1}{2} \pmod{2}$, we prove that

$$t(a, 3a, 8k + 4, 4m + 2; n) = \frac{2}{3} N(a, 3a, 8k + 4, 4m + 2; 8n + 4m + 8k + 4a + 6).$$

Let $a, k, m \in \mathbb{N}$ with $2 \nmid a$. We also state that

$$\begin{aligned} t(a, a, 2a, 4k; 4m + 3a) &= 4t(a, 2a, 4a, k; m), \quad t(a, a, 6a, 4k; 4m + 3a) = 2t(a, a, 6a, k; m), \\ t(a, a, 8a, 2k; 2m) &= t(a, 2a, 2a, k; m) \quad \text{and} \quad t(a, a, 8a, 2k; 2m + a) = 2t(a, 4a, 4a, k; m). \end{aligned}$$

In addition, we give explicit formulas for $t(1, 3, 3, 6; n)$, $t(1, 1, 8, 8; n)$ and $t(1, 1, 4, 8; n)$. We also pose many conjectures on the relations between $t(a, b, c, d; n)$ and $N(a, b, c, d; n)$.

2. Main results

Lemma 2.1. *Suppose $a, k, m, n \in \mathbb{N}$ and $2 \nmid a$. Then*

$$N(a, a, 2k, 2m; 2n) = N(a, a, k, m; n).$$

Proof. Suppose $|q| < 1$. Using (1.10) we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} N(a, a, 2k, 2m; n) q^n \\ &= \varphi(q^a)^2 \varphi(q^{2k}) \varphi(q^{2m}) = (\varphi(q^{2a})^2 + 4q^a \psi(q^{4a})^2) \varphi(q^{2k}) \varphi(q^{2m}). \end{aligned}$$

Extracting the even powers we obtain

$$\sum_{n=0}^{\infty} N(a, a, 2k, 2m; 2n) q^{2n} = \varphi(q^{2a})^2 \varphi(q^{2k}) \varphi(q^{2m}).$$

Replacing q with $q^{1/2}$ in the above formula we obtain

$$\sum_{n=0}^{\infty} N(a, a, 2k, 2m; 2n) q^n = \varphi(q^a)^2 \varphi(q^k) \varphi(q^m) = \sum_{n=0}^{\infty} N(a, a, k, m; n) q^n.$$

Comparing the coefficients of q^n on both sides we obtain the result.

Lemma 2.2. *For $|q| < 1$ we have*

$$\varphi(q)^3 = \varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^4)^2 + 12q^2\psi(q^4)^2\psi(q^8) + 8q^3\psi(q^8)^3.$$

Proof. By (1.8) and (1.9),

$$\begin{aligned} \varphi(q)^3 &= (\varphi(q^4) + 2q\psi(q^8))^3 \\ &= \varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^8)(\varphi(q^4) + 2q\psi(q^8)) + 8q^3\psi(q^8)^3 \\ &= \varphi(q^4)^3 + 6q\psi(q^4)^2(\varphi(q^4) + 2q\psi(q^8)) + 8q^3\psi(q^8)^3. \end{aligned}$$

This yields the result.

Theorem 2.1. *Let $a \in \{1, 3, 5, \dots\}$ and $m \in \{0, 1, 2, \dots\}$. For $n \in \mathbb{N}$ we have*

$$\begin{aligned} &t(a, a, 2a, 8m + 4; n) \\ &= \frac{2}{3}(N(a, a, a, 4m + 2; 4n + 4m + 2a + 2) - N(a, a, a, 4m + 2; n + m + (a + 1)/2)) \\ &= \frac{2}{3}(N(a, a, 2a, 8m + 4; 8n + 8m + 4a + 4) - N(a, a, 2a, 8m + 4; 2n + 2m + a + 1)). \end{aligned}$$

Proof. Suppose $|q| < 1$. By Lemma 2.2,

$$\begin{aligned} &\sum_{n=0}^{\infty} N(a, a, a, 4m + 2; n) q^n \\ (2.1) \quad &= \varphi(q^a)^3 \varphi(q^{4m+2}) \\ &= (\varphi(q^{4a})^3 + 6q^a \varphi(q^{4a}) \psi(q^{4a})^2 + 12q^{2a} \psi(q^{4a})^2 \psi(q^{8a}) + 8q^{3a} \psi(q^{8a})^3) \\ &\quad \times (\varphi(q^{4(4m+2)}) + 2q^{4m+2} \psi(q^{8(4m+2)})) \end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Hence from (2.1) we deduce that

$$\begin{aligned} &\sum_{n=0}^{\infty} N(a, a, a, 4m + 2; 4n) q^{4n} \\ &= \varphi(q^{4a})^3 \varphi(q^{16m+8}) + 12q^{2a} \psi(q^{4a})^2 \psi(q^{8a}) \cdot 2q^{4m+2} \psi(q^{32m+16}). \end{aligned}$$

Replacing q with $q^{1/4}$ in the above formula we obtain

$$\sum_{n=0}^{\infty} N(a, a, a, 4m + 2; 4n) q^n$$

$$\begin{aligned}
&= \varphi(q^a)^3 \varphi(q^{4m+2}) + 24q^{m+(a+1)/2} \psi(q^a)^2 \psi(q^{2a}) \psi(q^{8m+4}) \\
&= \sum_{n=0}^{\infty} N(a, a, a, 4m+2; n) q^n + 24q^{m+(a+1)/2} \sum_{n=0}^{\infty} t'(a, a, 2a, 8m+4; n) q^n.
\end{aligned}$$

Now comparing the coefficients of $q^{n+m+(a+1)/2}$ on both sides and then applying (1.1) and Lemma 2.1 we obtain

$$\begin{aligned}
&\frac{3}{2} t(a, a, 2a, 8m+4; n) \\
&= 24t'(a, a, 2a, 8m+4; n) \\
&= N(a, a, a, 4m+2; 4n+4m+2a+2) - N(a, a, a, 4m+2; n+m+(a+1)/2) \\
&= N(a, a, 2a, 8m+4; 8n+8m+4a+4) - N(a, a, 2a, 8m+4; 2n+2m+2a+2).
\end{aligned}$$

This is the result.

Lemma 2.3. For $|q| < 1$ we have

$$\begin{aligned}
\varphi(q)\varphi(q^3) &= \varphi(q^{16})\varphi(q^{48}) + 4q^{16}\psi(q^{32})\psi(q^{96}) + 2q\varphi(q^{48})\psi(q^8) + 2q^3\varphi(q^{16})\psi(q^{24}) \\
&\quad + 6q^4\psi(q^8)\psi(q^{24}) + 4q^{13}\psi(q^8)\psi(q^{96}) + 4q^7\psi(q^{24})\psi(q^{32}).
\end{aligned}$$

Proof. By (1.12),

$$\begin{aligned}
\varphi(q)\varphi(q^3) &= (\varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8))(\varphi(q^{48}) + 2q^{12}\psi(q^{96}) + 2q^3\psi(q^{24})) \\
&= \varphi(q^{16})\varphi(q^{48}) + 4q^{16}\psi(q^{32})\psi(q^{96}) + 2q\varphi(q^{48})\psi(q^8) + 2q^3\varphi(q^{16})\psi(q^{24}) \\
&\quad + 2q^4(\varphi(q^{48})\psi(q^{32}) + q^8\varphi(q^{16})\psi(q^{96}) + 2\psi(q^8)\psi(q^{24})) \\
&\quad + 4q^{13}\psi(q^8)\psi(q^{96}) + 4q^7\psi(q^{24})\psi(q^{32}).
\end{aligned}$$

Note that $\varphi(q^{48})\psi(q^{32}) + q^8\varphi(q^{16})\psi(q^{96}) = \psi(q^8)\psi(q^{24})$ by (1.11). We then obtain the result.

Theorem 2.2. Let $a \in \{1, 3, 5, \dots\}$, $k, m \in \{0, 1, 2, \dots\}$ and $k \equiv m \pmod{2}$. For $n \in \mathbb{N}$ we have

$$\begin{aligned}
&t(a, 3a, 4k+2, 4m+2; n) \\
&= \frac{2}{3} (N(a, 3a, 4k+2, 4m+2; 8n+4m+4k+4a+4) \\
&\quad - N(a, 3a, 4k+2, 4m+2; 2n+m+k+a+1)).
\end{aligned}$$

Proof. Suppose $|q| < 1$. Using Lemma 2.3 and (1.12) we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(a, 3a, 4k+2, 4m+2; n) q^n \\
&= \varphi(q^a)\varphi(q^{3a})\varphi(q^{4k+2})\varphi(q^{4m+2}) \\
&= \varphi(q^a)\varphi(q^{3a})(\varphi(q^{4(4k+2)}) + 2q^{4k+2}\psi(q^{8(4k+2)}))(\varphi(q^{4(4m+2)}) + 2q^{4m+2}\psi(q^{8(4m+2)})) \\
&= (\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) + 2q^a\varphi(q^{48a})\psi(q^{8a}) + 2q^{3a}\varphi(q^{16a})\psi(q^{24a}) \\
&\quad + 6q^{4a}\psi(q^{8a})\psi(q^{24a}) + 4q^{13a}\psi(q^{8a})\psi(q^{96a}) + 4q^{7a}\psi(q^{24a})\psi(q^{32a})) \\
&\quad \times (\varphi(q^{16k+8})\varphi(q^{16m+8}) + 2q^{4m+2}\varphi(q^{16k+8})\psi(q^{32m+16}) \\
&\quad + 2q^{4k+2}\psi(q^{32k+16})\varphi(q^{16m+8}) + 4q^{4k+4m+4}\psi(q^{32k+16})\psi(q^{32m+16})).
\end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above we deduce that

$$\begin{aligned} & \sum_{n=0}^{\infty} N(a, 3a, 4k+2, 4m+2; 8n) q^{8n} \\ &= (\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}))\varphi(q^{16k+8})\varphi(q^{16m+8}) \\ & \quad + 24q^{4(k+m+a+1)}\psi(q^{8a})\psi(q^{24a})\psi(q^{32k+16})\psi(q^{32m+16}). \end{aligned}$$

Replacing q with $q^{1/8}$ in the above formula we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} N(a, 3a, 4k+2, 4m+2; 8n) q^n \\ &= (\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a}))\varphi(q^{2k+1})\varphi(q^{2m+1}) \\ & \quad + 24q^{(k+m+a+1)/2}\psi(q^a)\psi(q^{3a})\psi(q^{4k+2})\psi(q^{4m+2}). \end{aligned}$$

On the other hand, using (1.9) we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} N(a, 3a, 4k+2, 4m+2; n) q^n \\ &= \varphi(q^a)\varphi(q^{3a})\varphi(q^{4k+2})\varphi(q^{4m+2}) \\ &= (\varphi(q^{4a}) + 2q^a\psi(q^{8a}))(\varphi(q^{12a}) + 2q^{3a}\psi(q^{24a}))\varphi(q^{4k+2})\varphi(q^{4m+2}). \end{aligned}$$

Extracting the even powers we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} N(a, 3a, 4k+2, 4m+2; 2n) q^{2n} \\ &= (\varphi(q^{4a})\varphi(q^{12a}) + 4q^{4a}\psi(q^{8a})\psi(q^{24a}))\varphi(q^{4k+2})\varphi(q^{4m+2}). \end{aligned}$$

Replacing q with $q^{1/2}$ we then obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} N(a, 3a, 4k+2, 4m+2; 2n) q^n \\ &= (\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a}))\varphi(q^{2k+1})\varphi(q^{2m+1}). \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{n=0}^{\infty} (N(a, 3a, 4k+2, 4m+2; 8n) - N(a, 3a, 4k+2, 4m+2; 2n)) q^n \\ &= 24q^{(k+m+a+1)/2}\psi(q^a)\psi(q^{3a})\psi(q^{4k+2})\psi(q^{4m+2}). \\ &= 24q^{(k+m+a+1)/2} \sum_{n=0}^{\infty} t'(a, 3a, 4k+2, 4m+2; n) q^n \\ &= \frac{3}{2}q^{(k+m+a+1)/2} \sum_{n=0}^{\infty} t(a, 3a, 4k+2, 4m+2; n) q^n. \end{aligned}$$

Comparing the coefficients of $q^{n+(k+m+a+1)/2}$ yields the result.

Theorem 2.3. *Let $a, k \in \mathbb{N}$ with $2 \nmid ak$. For $m \in \mathbb{N}$ we have*

$$t(a, 3a, k, k; m) = \frac{2}{3}N(a, 3a, 2k, 2k; 8m + 4a + 2k).$$

Proof. By (1.10),

$$\varphi(q^{2k})^2 = \varphi(q^{4k})^2 + 4q^{2k}\psi(q^{8k})^2 = \varphi(q^{8k})^2 + 4q^{4k}\psi(q^{16k})^2 + 4q^{2k}\psi(q^{8k})^2.$$

Thus, applying Lemma 2.3 we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} N(a, 3a, 2k, 2k; n)q^n \\ &= \varphi(q^a)\varphi(q^{3a})\varphi(q^{2k})^2 \\ &= (\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) + 2q^a\varphi(q^{48a})\psi(q^{8a}) + 2q^{3a}\varphi(q^{16a})\psi(q^{24a}) \\ &\quad + 6q^{4a}\psi(q^{8a})\psi(q^{24a}) + 4q^{13a}\psi(q^{8a})\psi(q^{96a}) + 4q^{7a}\psi(q^{24a})\psi(q^{32a})) \\ &\quad \times (\varphi(q^{8k})^2 + 4q^{4k}\psi(q^{16k})^2 + 4q^{2k}\psi(q^{8k})^2). \end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^s)^r$ and $\psi(q^s)^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above we deduce that

$$\begin{aligned} & \sum_{m=0}^{\infty} N(a, 3a, 2k, 2k; 8m + 4a + 2k)q^{8m+4a+2k} \\ &= 6q^{4a}\psi(q^{8a})\psi(q^{24a}) \cdot 4q^{2k}\psi(q^{8k})^2 = 24q^{4a+2k}\psi(q^{8a})\psi(q^{24a})\psi(q^{8k})^2. \end{aligned}$$

Replacing q with $q^{1/8}$ we get

$$\begin{aligned} & \sum_{m=0}^{\infty} N(a, 3a, 2k, 2k; 8m + 4a + 2k)q^m \\ &= 24\psi(q^a)\psi(q^{3a})\psi(q^k)^2 = 24 \sum_{m=0}^{\infty} t'(a, 3a, k, k; m)q^m = \frac{24}{16} \sum_{m=0}^{\infty} t(a, 3a, k, k; m)q^m. \end{aligned}$$

Comparing the coefficients of q^m on both sides yields the result.

Corollary 2.1. *For $n \in \mathbb{N}$ we have*

$$N(2, 2, 3, 9; 8n + 6) = \frac{3}{5}N(1, 1, 3, 9; 8n + 6).$$

Proof. Taking $a = 3$ and $k = 1$ in Theorem 2.3 we see that $t(1, 1, 3, 9; m) = \frac{2}{3}N(2, 2, 3, 9; 8m + 14)$. On the other hand, from [WS, the proof of Theorem 2.3] we know that $t(1, 1, 3, 9; m) = \frac{2}{5}N(1, 1, 3, 9; 8m + 14)$. Thus, the result follows.

Theorem 2.4. *Let $a \in \{1, 3, 5, \dots\}$, $k, m \in \{0, 1, 2, \dots\}$ and $n \in \mathbb{N}$. If $n \equiv k + \frac{a-1}{2} \pmod{2}$, then*

$$t(a, 3a, 8k + 4, 4m + 2; n) = \frac{2}{3}N(a, 3a, 8k + 4, 4m + 2; 8n + 4m + 8k + 4a + 6).$$

Proof. Suppose $|q| < 1$. Using Lemma 2.3 and (1.12) we see that

$$\begin{aligned}
& \sum_{n=0}^{\infty} N(a, 3a, 8k+4, 4m+2; n) q^n \\
&= \varphi(q^a) \varphi(q^{3a}) \varphi(q^{8k+4}) \varphi(q^{4m+2}) \\
&= \varphi(q^a) \varphi(q^{3a}) (\varphi(q^{4(8k+4)}) + 2q^{8k+4} \psi(q^{8(8k+4)})) (\varphi(q^{4(4m+2)}) + 2q^{4m+2} \psi(q^{8(4m+2)})) \\
&= (\varphi(q^{16a}) \varphi(q^{48a}) + 4q^{16a} \psi(q^{32a}) \psi(q^{96a}) + 2q^a \varphi(q^{48a}) \psi(q^{8a}) + 2q^{3a} \varphi(q^{16a}) \psi(q^{24a}) \\
&\quad + 6q^{4a} \psi(q^{8a}) \psi(q^{24a}) + 4q^{13a} \psi(q^{8a}) \psi(q^{96a}) + 4q^{7a} \psi(q^{24a}) \psi(q^{32a})) \\
&\quad \times (\varphi(q^{32k+16}) \varphi(q^{16m+8}) + 2q^{4m+2} \varphi(q^{32k+16}) \psi(q^{32m+16}) \\
&\quad + 2q^{8k+4} \psi(q^{64k+32}) \varphi(q^{16m+8}) + 4q^{8k+4m+6} \psi(q^{64k+32}) \psi(q^{32m+16})).
\end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above and the fact that $4m+2 \equiv 4 - 2(-1)^m \pmod{8}$ we deduce that

$$\begin{aligned}
& \sum_{n=0}^{\infty} N(a, 3a, 8k+4, 4m+2; 8n+4-2(-1)^m) q^{8n+4-2(-1)^m} \\
&= (\varphi(q^{16a}) \varphi(q^{48a}) + 4q^{16a} \psi(q^{32a}) \psi(q^{96a})) \cdot 2q^{4m+2} \varphi(q^{32k+16}) \psi(q^{32m+16}) \\
&\quad + 6q^{4a} \psi(q^{8a}) \psi(q^{24a}) \cdot 4q^{8k+4m+6} \psi(q^{64k+32}) \psi(q^{32m+16})
\end{aligned}$$

and so

$$\begin{aligned}
& \sum_{n=0}^{\infty} N(a, 3a, 8k+4, 4m+2; 8n+4-2(-1)^m) q^{8n} \\
&= 2q^{8[m/2]} \varphi(q^{32k+16}) \psi(q^{32m+16}) (\varphi(q^{16a}) \varphi(q^{48a}) + 4q^{16a} \psi(q^{32a}) \psi(q^{96a})) \\
&\quad + 24q^{8(k+[m/2]+(a+1)/2)} \psi(q^{8a}) \psi(q^{24a}) \psi(q^{64k+32}) \psi(q^{32m+16}),
\end{aligned}$$

where $[x]$ is the greatest integer not exceeding x . Replacing q with $q^{1/8}$ in the above formula we obtain

$$\begin{aligned}
& \sum_{n=0}^{\infty} N(a, 3a, 8k+4, 4m+2; 8n+4-2(-1)^m) q^n \\
&= 2q^{8[m/2]} \varphi(q^{4k+2}) \psi(q^{4m+2}) (\varphi(q^{2a}) \varphi(q^{6a}) + 4q^{2a} \psi(q^{4a}) \psi(q^{12a})) \\
&\quad + 24q^{k+[m/2]+(a+1)/2} \psi(q^a) \psi(q^{3a}) \psi(q^{8k+4}) \psi(q^{4m+2}),
\end{aligned}$$

Suppose that $n \equiv k + (a-1)/2 \pmod{2}$. Then $n+k+[m/2]+(a+1)/2 \equiv [m/2]+1 \pmod{2}$. Now comparing the coefficients of $q^{n+k+[m/2]+(a+1)/2}$ in the above expansion we obtain

$$\begin{aligned}
& N(a, 3a, 8k+4, 4m+2; 8(n+k+[m/2]+(a+1)/2)+4-2(-1)^m) \\
&= 24t'(a, 3a, 8k+4, 4m+2; n) = \frac{3}{2} t(a, 3a, 8k+4, 4m+2; n).
\end{aligned}$$

This yields the result.

Theorem 2.5. *Let $a, k \in \mathbb{N}$ with $2 \nmid a$. For $m \in \mathbb{N}$ we have*

$$t(a, a, 6a, 4k; 4m + 3a) = 2t(a, a, 6a, k; m).$$

Proof. Suppose $|q| < 1$. Using (1.8)-(1.12) we see that

$$\begin{aligned}
 (2.2) \quad & \sum_{n=0}^{\infty} t'(a, a, 6a, 4k; n) q^n \\
 &= \psi(q^a)^2 \psi(q^{6a}) \psi(q^{4k}) = \varphi(q^a) \psi(q^{2a}) \psi(q^{6a}) \psi(q^{4k}) \\
 &= (\varphi(q^{4a}) + 2q^a \psi(q^{8a})) (\varphi(q^{12a}) \psi(q^{8a}) + q^{2a} \varphi(q^{4a}) \psi(q^{24a})) \psi(q^{4k}) \\
 &= (\varphi(q^{4a}) \varphi(q^{12a}) \psi(q^{8a}) + 2q^a \psi(q^{8a})^2 \varphi(q^{12a}) + q^{2a} \varphi(q^{4a})^2 \psi(q^{24a}) \\
 &\quad + 2q^{3a} \varphi(q^{4a}) \psi(q^{8a}) \psi(q^{24a})) \psi(q^{4k}).
 \end{aligned}$$

Extracting the powers of q^{4m+3a} we get

$$\begin{aligned}
 & \sum_{m=0}^{\infty} t'(a, a, 6a, 4k; 4m + 3a) q^{4m+3a} \\
 &= 2q^{3a} \varphi(q^{4a}) \psi(q^{8a}) \psi(q^{24a}) \psi(q^{4k}) = 2q^{3a} \psi(q^{4a})^2 \psi(q^{24a}) \psi(q^{4k}).
 \end{aligned}$$

Replacing q with $q^{1/4}$ we deduce that

$$\sum_{m=0}^{\infty} t'(a, a, 6a, 4k; 4m + 3a) q^m = 2\psi(q^a)^2 \psi(q^{6a}) \psi(q^k) = 2 \sum_{m=0}^{\infty} t'(a, a, 6a, k; m) q^m.$$

Hence

$$\begin{aligned}
 t(a, a, 6a, 4k; 4m + 3a) &= 16t'(a, a, 6a, 4k; 4m + 3a) = 32t'(a, a, 6a, k; m) \\
 &= 2t(a, a, 6a, k; m)
 \end{aligned}$$

as asserted.

Theorem 2.6. *Let $a, k \in \mathbb{N}$ with $2 \nmid a$. For $m \in \mathbb{N}$ we have*

$$t(a, a, 2a, 4k; 4m + 3a) = 4t(a, 2a, 4a, k; m).$$

Proof. Suppose $|q| < 1$. Using (1.8) and (1.12) we see that

$$\begin{aligned}
 & \sum_{n=0}^{\infty} t'(a, a, 2a, 4k; n) q^n \\
 &= \psi(q^a)^2 \psi(q^{2a}) \psi(q^{4k}) = \varphi(q^a) \psi(q^{2a})^2 \psi(q^{4k}) \\
 &= \varphi(q^a) \varphi(q^{2a}) \psi(q^{4a}) \psi(q^{4k}) \\
 &= (\varphi(q^{4a}) + 2q^a \psi(q^{8a})) (\varphi(q^{8a}) + 2q^{2a} \psi(q^{16a})) \psi(q^{4a}) \psi(q^{4k}) \\
 &= (\varphi(q^{4a}) \varphi(q^{8a}) + 2q^a \varphi(q^{8a}) \psi(q^{8a}) + 2q^{2a} \varphi(q^{4a}) \psi(q^{16a}) \\
 &\quad + 4q^{3a} \psi(q^{8a}) \psi(q^{16a})) \psi(q^{4a}) \psi(q^{4k}).
 \end{aligned}$$

Extracting the powers of q^{4m+3a} we get

$$\sum_{m=0}^{\infty} t'(a, a, 2a, 4k; 4m + 3a) q^{4m+3a} = 4q^{3a} \psi(q^{8a}) \psi(q^{16a}) \psi(q^{4a}) \psi(q^{4k}).$$

Replacing q with $q^{1/4}$ we see that

$$\begin{aligned} & \sum_{m=0}^{\infty} t'(a, a, 2a, 4k; 4m + 3a) q^m \\ &= 4\psi(q^a)\psi(q^{2a})\psi(q^{4a})\psi(q^k) = 4 \sum_{m=0}^{\infty} t'(a, 2a, 4a, k; m) q^m. \end{aligned}$$

Hence

$$\begin{aligned} t(a, a, 2a, 4k; 4m + 3a) &= 16t'(a, a, 2a, 4k; 4m + 3a) = 64t'(a, 2a, 4a, k; m) \\ &= 4t(a, 2a, 4a, k; m), \end{aligned}$$

which completes the proof.

Theorem 2.7. *Let $a, k \in \mathbb{N}$ with $2 \nmid a$. For $n \in \mathbb{N}$ we have*

$$t(a, a, 8a, 2k; 2n) = t(a, 2a, 2a, k; n)$$

and

$$t(a, a, 8a, 2k; 2n + a) = 2t(a, 4a, 4a, k; n).$$

Proof. Suppose $|q| < 1$. Using (1.8) and (1.9) we see that

$$\begin{aligned} (2.3) \quad \sum_{n=0}^{\infty} t'(a, a, 8a, 2k; n) q^n &= \psi(q^a)^2 \psi(q^{8a}) \psi(q^{2k}) = \varphi(q^a) \psi(q^{2a}) \psi(q^{8a}) \psi(q^{2k}) \\ &= \psi(q^{2a}) (\varphi(q^{4a}) + 2q^a \psi(q^{8a})) \psi(q^{8a}) \psi(q^{2k}). \end{aligned}$$

Extracting the even powers we see that

$$\sum_{n=0}^{\infty} t'(a, a, 8a, 2k; 2n) q^{2n} = \psi(q^{2a}) \varphi(q^{4a}) \psi(q^{8a}) \psi(q^{2k}) = \psi(q^{2a}) \psi(q^{4a})^2 \psi(q^{2k}).$$

Replacing q with $q^{1/2}$ we then get

$$\sum_{n=0}^{\infty} t'(a, a, 8a, 2k; 2n) q^n = \psi(q^a) \psi(q^{2a})^2 \psi(q^k) = \sum_{n=0}^{\infty} t'(a, 2a, 2a, k; n) q^n.$$

Hence

$$t(a, a, 8a, 2k; 2n) = 16t'(a, a, 8a, 2k; 2n) = 16t'(a, 2a, 2a, k; n) = t(a, 2a, 2a, k; n).$$

On the other hand, extracting the odd powers in (2.3) we see that

$$\sum_{n=0}^{\infty} t'(a, a, 8a, 2k; 2n + a) q^{2n+a} = \psi(q^{2a}) \cdot 2q^a \psi(q^{8a}) \psi(q^{8a}) \psi(q^{2k}).$$

Replacing q with $q^{1/2}$ we then get

$$\sum_{n=0}^{\infty} t'(a, a, 8a, 2k; 2n + a) q^n = 2\psi(q^a) \psi(q^{4a})^2 \psi(q^k) = \sum_{n=0}^{\infty} 2t'(a, 4a, 4a, k; n) q^n.$$

Hence

$$t(a, a, 8a, 2k; 2n + a) = 16t'(a, a, 8a, 2k; 2n + a) = 32t'(a, 4a, 4a, k; n) = 2t(a, 4a, 4a, k; n).$$

We are done.

Theorem 2.8. *For $n \in \mathbb{N}$ we have*

$$t(1, 1, 8, 8; n) = \sigma(4n + 9) - (2 - (-1)^n) \sum_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z}, x \equiv 1 \pmod{4} \\ 4n+9=x^2+4y^2}} x.$$

Proof. By Theorem 2.7, $t(1, 1, 8, 8; n) = t(1, 2, 2, 4; n/2)$ for even n , and $t(1, 1, 8, 8; n) = 2t(1, 4, 4, 4; (n-1)/2)$ for odd n . Now applying [WS, Theorems 3.2 and 3.3] yields the result.

Theorem 2.9. *For $n \in \mathbb{N}$ we have*

$$t(1, 1, 4, 8; n) = 2(-1)^n \sum_{d|4n+7} d \left(\frac{2}{d} \right) - (1 - (-1)^n) \sum_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z}, x \equiv 1 \pmod{4} \\ 4n+7=x^2+2y^2}} x.$$

Proof. By Theorem 2.7, $t(1, 1, 4, 8; n) = t(1, 2, 2, 2; n/2)$ for even n , and $t(1, 1, 4, 8; n) = 2t(1, 2, 4, 4; (n-1)/2)$ for odd n . It is easily seen that

$$\begin{aligned} t(1, 2, 2, 2; m) &= \left| \left\{ (x, y, z, w) \in \mathbb{Z}^4 \mid 8m + 7 = x^2 + 2y^2 + 2z^2 + 2w^2, 2 \nmid xyzw \right\} \right| \\ &= \left| \left\{ (x, y, z, w) \in \mathbb{Z}^4 \mid 8m + 7 = x^2 + 2y^2 + 2z^2 + 2w^2 \right\} \right| \\ &= N(1, 2, 2, 2; m). \end{aligned}$$

Now applying [W2, (1.4)] and [WS, Theorem 3.4] yields the result.

Lemma 2.4. *For $|q| < 1$ we have*

$$\varphi(q)^2 = \varphi(q^8)^2 + 4q^4\psi(q^{16})^2 + 4q^2\psi(q^8)^2 + 4q\varphi(q^{16})\psi(q^8) + 8q^5\psi(q^8)\psi(q^{32}).$$

Proof. By (1.10),

$$\begin{aligned} \varphi(q)^2 - \varphi(q^8)^2 &= (\varphi(q)^2 - \varphi(q^2)^2) + (\varphi(q^2)^2 - \varphi(q^4)^2) + (\varphi(q^4)^2 - \varphi(q^8)^2) \\ &= 4q\psi(q^4)^2 + 4q^2\psi(q^8)^2 + 4q^4\psi(q^{16})^2. \end{aligned}$$

By (1.8) and (1.9),

$$\psi(q^4)^2 = \varphi(q^4)\psi(q^8) = (\varphi(q^{16}) + 2q^4\psi(q^{32}))\psi(q^8).$$

Now combining the above we deduce the result.

Theorem 2.10. *If $n \in \mathbb{N}$ and $8n + 13 = 3^\beta n_1$ with $n_1 \in \mathbb{N}$ and $3 \nmid n_1$, then*

$$\begin{aligned} t(1, 3, 3, 6; n) &= \frac{2}{5} N(1, 3, 3, 6; 8n + 13) \\ &= \frac{2}{3} \left(3^\beta + \left(\frac{n_1}{3} \right) \right) \prod_{p|n_1} \frac{p^{\text{ord}_p n_1 + 1} - \left(\frac{6}{p} \right)^{\text{ord}_p n_1 + 1}}{p - \left(\frac{6}{p} \right)}, \end{aligned}$$

where p runs all distinct prime divisors of n_1 , $\left(\frac{a}{p} \right)$ is the Legendre symbol and $\text{ord}_p n_1$ is the unique nonnegative integer r such that $p^r \mid n_1$ but $p^{r+1} \nmid n_1$.

Proof. Suppose $|q| < 1$. By (1.12) and Lemma 2.4,

$$\begin{aligned}
& \sum_{n=0}^{\infty} N(1, 3, 3, 6; n) q^n \\
&= \varphi(q) \varphi(q^6) \varphi(q^3)^2 \\
&= (\varphi(q^{16}) + 2q^4 \psi(q^{32}) + 2q \psi(q^8)) (\varphi(q^{24}) + 2q^6 \psi(q^{48})) \\
&\quad \times (\varphi(q^{24})^2 + 4q^{12} \psi(q^{48})^2 + 4q^6 \psi(q^{24})^2 + 4q^3 \varphi(q^{48}) \psi(q^{24}) + 8q^{15} \psi(q^{24}) \psi(q^{96})) \\
&= (\varphi(q^{16}) \varphi(q^{24}) + 2q^4 \psi(q^{32}) \varphi(q^{24}) + 2q \psi(q^8) \varphi(q^{24}) \\
&\quad + 2q^6 \varphi(q^{16}) \psi(q^{48}) + 4q^{10} \psi(q^{32}) \psi(q^{48}) + 4q^7 \psi(q^8) \psi(q^{48})) \\
&\quad \times (\varphi(q^{24})^2 + 4q^{12} \psi(q^{48})^2 + 4q^6 \psi(q^{24})^2 + 4q^3 \varphi(q^{48}) \psi(q^{24}) + 8q^{15} \psi(q^{24}) \psi(q^{96})).
\end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^s)^r$ and $\psi(q^s)^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above we deduce that

$$\begin{aligned}
& \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 13) q^{8n+13} \\
&= \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 5) q^{8n+5} \\
&= 2q \psi(q^8) \varphi(q^{24}) \cdot 4q^{12} \psi(q^{48})^2 + 2q^6 \varphi(q^{16}) \psi(q^{48}) \cdot 8q^{15} \psi(q^{24}) \psi(q^{96}) \\
&\quad + 4q^{10} \psi(q^{32}) \psi(q^{48}) \cdot 4q^3 \psi(q^{24}) \varphi(q^{48}) + 4q^7 \psi(q^8) \psi(q^{48}) \cdot 4q^6 \psi(q^{24})^2
\end{aligned}$$

and so

$$\begin{aligned}
& \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 13) q^{8n} \\
&= 8\psi(q^8) \varphi(q^{24}) \varphi(q^{48}) \psi(q^{96}) + 16q^8 \varphi(q^{16}) \psi(q^{48}) \psi(q^{24}) \psi(q^{96}) \\
&\quad + 16\psi(q^{24}) \psi(q^{32}) \psi(q^{48}) \varphi(q^{48}) + 16\psi(q^8) \psi(q^{48}) \psi(q^{24})^2.
\end{aligned}$$

By (1.8),

$$\varphi(q^{24}) \varphi(q^{48}) \psi(q^{96}) = \frac{\psi(q^{24})^2}{\psi(q^{48})} \cdot \frac{\psi(q^{48})^2}{\psi(q^{96})} \cdot \psi(q^{96}) = \psi(q^{24})^2 \psi(q^{48}).$$

By (1.11),

$$\psi(q^{32}) \varphi(q^{48}) + q^8 \varphi(q^{16}) \psi(q^{96}) = \psi(q^8) \psi(q^{24}).$$

Hence,

$$\begin{aligned}
& \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 13) q^{8n} \\
&= 8\psi(q^8) \psi(q^{24})^2 \psi(q^{48}) + 16\psi(q^{24}) \psi(q^{48}) \psi(q^8) \psi(q^{24}) + 16\psi(q^8) \psi(q^{48}) \psi(q^{24})^2 \\
&= 40\psi(q^8) \psi(q^{24})^2 \psi(q^{48}).
\end{aligned}$$

Replacing q with $q^{1/8}$ we see that

$$\sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 13) q^n$$

$$= 40\psi(q)\psi(q^3)^2\psi(q^6) = 40 \sum_{n=0}^{\infty} t'(1, 3, 3, 6; n)q^n = \frac{5}{2} \sum_{n=0}^{\infty} t(1, 3, 3, 6; n)q^n.$$

Hence

$$t(1, 3, 3, 6; n) = \frac{2}{5}N(1, 3, 3, 6; 8n + 13).$$

Now applying [AW, Theorem 4.1] yields the remaining part.

Remark 2.1 Using Maple we find

$$t(a, b, c, d; n) = \frac{2}{5}N(a, b, c, d; 8n + a + b + c + d)$$

for $(a, b, c, d) = (1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 2, 3), (1, 1, 3, 9), (1, 3, 3, 3), (1, 3, 3, 6)$ and $(1, 3, 9, 9)$. The cases $(1, 1, 3, 9)$ and $(1, 3, 9, 9)$ were solved by Wang and Sun in [WS].

Theorem 2.11. *If $n \in \mathbb{N}$ and $n \equiv 1, 2 \pmod{4}$, then*

$$t(1, 1, 4, 6; n) = 2N(1, 1, 4, 6; 2n + 3).$$

Proof. By (1.12) and Lemma 2.4,

$$\begin{aligned} (2.4) \quad & \sum_{n=0}^{\infty} N(1, 1, 4, 6; n)q^n \\ &= \varphi(q)^2\varphi(q^4)\varphi(q^6) \\ &= \varphi(q)^2(\varphi(q^{16}) + 2q^4\psi(q^{32}))(\varphi(q^{24}) + 2q^6\psi(q^{48})) \\ &= (\varphi(q^8)^2 + 4q^4\psi(q^{16})^2 + 4q^2\psi(q^8)^2 + 4q\varphi(q^{16})\psi(q^8) + 8q^5\psi(q^8)\psi(q^{32})) \\ & \quad \times (\varphi(q^{16})\varphi(q^{24}) + 4q^{10}\psi(q^{32})\psi(q^{48}) + 2q^4\varphi(q^{24})\psi(q^{32}) + 2q^6\varphi(q^{16})\psi(q^{48})). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 5)q^{8m+5} &= 16q^5\psi(q^8)\psi(q^{32})\varphi(q^{16})\varphi(q^{24}), \\ \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 7)q^{8m+7} &= 8q^7\psi(q^8)\psi(q^{48})(\varphi(q^{16})^2 + 4q^8\psi(q^{32})^2). \end{aligned}$$

From the above and (1.10) we deduce that

$$\begin{aligned} (2.5) \quad & \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 5)q^m = 16\psi(q)\psi(q^4)\varphi(q^2)\varphi(q^3) = 16\psi(q)\psi(q^2)^2\varphi(q^3), \\ & \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 7)q^m = 8\psi(q)\psi(q^6)(\varphi(q^2)^2 + 4q\psi(q^4)^2) = 8\varphi(q)^2\psi(q)\psi(q^6). \end{aligned}$$

On the other hand, from (2.2) we know that

$$\sum_{m=0}^{\infty} t'(1, 1, 4, 6; 4m + 1)q^{4m+1} = 2q\psi(q^4)\psi(q^8)^2\varphi(q^{12}),$$

$$\sum_{m=0}^{\infty} t'(1, 1, 4, 6; 4m+2)q^{4m+2} = q^2\varphi(q^4)^2\psi(q^4)\psi(q^{24}).$$

Therefore,

$$\begin{aligned}\sum_{m=0}^{\infty} t'(1, 1, 4, 6; 4m+1)q^m &= 2\psi(q)\psi(q^2)^2\varphi(q^3), \\ \sum_{m=0}^{\infty} t'(1, 1, 4, 6; 4m+2)q^m &= \varphi(q)^2\psi(q)\psi(q^6).\end{aligned}$$

This together with (2.5) yields $N(1, 1, 4, 6; 8m+5) = 8t'(1, 1, 4, 6; 4m+1)$ and $N(1, 1, 4, 6; 8m+7) = 8t'(1, 1, 4, 6; 4m+2)$. To complete the proof, we recall that $t(a, b, c, d; n) = 16t'(a, b, c, d; n)$.

Theorem 2.12. *For $n \in \mathbb{N}$ with $n \equiv 1 \pmod{4}$ we have*

$$t(2, 2, 3, 9; n) = \frac{4}{3}N(2, 2, 3, 9; 2n+4).$$

Proof. Using (1.8)-(1.12) we see that for $|q| < 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} t'(2, 2, 3, 9; n)q^n \\ &= \psi(q^2)^2\psi(q^3)\psi(q^9) = \varphi(q^2)\psi(q^4)(\varphi(q^{18})\psi(q^{12}) + q^3\varphi(q^6)\psi(q^{36})) \\ &= (\varphi(q^8) + 2q^2\psi(q^{16}))\psi(q^4)((\varphi(q^{72}) + 2q^{18}\psi(q^{144}))\psi(q^{12}) \\ &\quad + q^3(\varphi(q^{24}) + 2q^6\psi(q^{48}))\psi(q^{36})).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{m=0}^{\infty} t'(2, 2, 3, 9; 4m+5)q^{4m+5} \\ &= (2q^5\varphi(q^{24})\psi(q^{16}) + 2q^9\varphi(q^8)\psi(q^{48}))\psi(q^4)\psi(q^{36}) \\ &= 2q^5\psi(q^4)\psi(q^{12})\psi(q^4)\psi(q^{36})\end{aligned}$$

and so

$$\sum_{m=0}^{\infty} t'(2, 2, 3, 9; 4m+5)q^m = 2\psi(q)^2\psi(q^3)\psi(q^9) = 2\sum_{m=0}^{\infty} t'(1, 1, 3, 9; m)q^m.$$

Therefore,

$$t(2, 2, 3, 9; 4m+5) = 16t'(2, 2, 3, 9; 4m+5) = 32t'(1, 1, 3, 9; m) = 2t(1, 1, 3, 9; m).$$

Taking $a = 3$ and $k = 1$ in Theorem 2.3 we see that $t(1, 1, 3, 9; m) = \frac{2}{3}N(2, 2, 3, 9; 8m+6)$. Thus, $t(2, 2, 3, 9; 4m+5) = 2t(1, 1, 3, 9; m) = \frac{4}{3}N(2, 2, 3, 9; 8m+14)$. This yields the result.

Theorem 2.13. *For $n \in \mathbb{N}$ we have*

$$t(1, 2, 2, 6; n) = \frac{1}{2}N(1, 1, 4, 6; 8n+11)$$

and

$$t(1, 1, 8, 12; 2n) = \frac{1}{2}N(1, 1, 8, 12; 16n+22).$$

Proof. From (2.3) and (1.8) we know that

$$\begin{aligned} & \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m+3) q^{8m+3} \\ &= 32q^{11} \varphi(q^{16}) \psi(q^8) \psi(q^{32}) \psi(q^{48}) = 32q^{11} \psi(q^8) \psi(q^{16})^2 \psi(q^{48}). \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m+3) q^m \\ &= 32q \psi(q) \psi(q^2)^2 \psi(q^6) = 32 \sum_{n=0}^{\infty} t'(1, 2, 2, 6; n) q^{n+1} = 2 \sum_{n=0}^{\infty} t(1, 2, 2, 6; n) q^{n+1}. \end{aligned}$$

Now comparing the coefficients of q^{n+1} we obtain $N(1, 1, 4, 6; 8n+11) = 2t(1, 2, 2, 6; n)$. Applying Lemma 2.1 and Theorem 2.7 we see that

$$t(1, 1, 8, 12; 2n) = t(1, 2, 2, 6; n) = \frac{1}{2} N(1, 1, 4, 6; 8n+11) = \frac{1}{2} N(1, 1, 8, 12; 16n+22).$$

We are done.

Theorem 2.14. *For $m \in \mathbb{N}$ we have*

$$t(1, 1, 6, 24; 4m+1) = 2t(2, 2, 3, 3; m)$$

and

$$t(1, 1, 6, 24; 4m) = t(1, 1, 3, 3; m) = 2^{\alpha+4} \sigma(m_1),$$

where α and m_1 are given by $m+1 = 2^\alpha 3^\beta m_1$ and $\gcd(m_1, 6) = 1$.

Proof. Taking $a = 1$ and $k = 6$ in (2.2) we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} t'(1, 1, 6, 24; n) q^n \\ &= (\varphi(q^4) \varphi(q^{12}) \psi(q^8) + 2q \psi(q^8)^2 \varphi(q^{12}) + q^2 \varphi(q^4)^2 \psi(q^{24}) \\ & \quad + 2q^3 \varphi(q^4) \psi(q^8) \psi(q^{24})) \psi(q^{24}). \end{aligned}$$

Hence, using (1.8) we see that

$$\sum_{m=0}^{\infty} t'(1, 1, 6, 24; 4m) q^{4m} = \varphi(q^4) \psi(q^8) \varphi(q^{12}) \psi(q^{24}) = \psi(q^4)^2 \psi(q^{12})^2$$

and

$$\sum_{m=0}^{\infty} t'(1, 1, 6, 24; 4m+1) q^{4m+1} = 2q \psi(q^8)^2 \varphi(q^{12}) \psi(q^{24}) = 2q \psi(q^8)^2 \psi(q^{12})^2.$$

Therefore,

$$\sum_{m=0}^{\infty} t'(1, 1, 6, 24; 4m) q^m = \psi(q)^2 \psi(q^3)^2 = \sum_{m=0}^{\infty} t'(1, 1, 3, 3; m) q^m,$$

$$\sum_{m=0}^{\infty} t'(1, 1, 6, 24; 4m+1)q^m = 2\psi(q^2)^2\psi(q^3)^2 = 2 \sum_{m=0}^{\infty} t'(2, 2, 3, 3; m)q^m.$$

Since $t(a, b, c, d; n) = 16t'(a, b, c, d; n)$, we obtain $t(1, 1, 6, 24; 4m+1) = 2t(2, 2, 3, 3; m)$ and $t(1, 1, 6, 24; 4m) = t(1, 1, 3, 3; m)$. Now applying [WS, Lemma 4.1] we deduce the theorem.

In conclusion, we pose the following conjectures.

Conjecture 2.1. *Let $n \in \mathbb{N}$ with $n \equiv 0, 3 \pmod{4}$. Then*

$$t(1, 1, 4, 6; n) = \frac{2}{3}N(1, 1, 4, 6; 8n+12) - N(1, 1, 4, 6; 2n+3).$$

Conjecture 2.2. *Let $n \in \mathbb{N}$. If $3 \mid n$, then*

$$t(1, 1, 8, 12; n) = \frac{1}{2}N(1, 1, 8, 12; 8n+22).$$

Conjecture 2.3. *Let $m \in \mathbb{N}$. Then*

$$t(1, 3, 8, 8; 3m) = \frac{1}{3}N(1, 3, 8, 8; 24m+20) - 2N(1, 3, 8, 8; 6m+5).$$

Conjecture 2.4. *Let $n \in \mathbb{N}$ with $n \equiv 0 \pmod{6}$. Then*

$$t(1, 2, 3, 8; n) = \frac{2}{3}N(1, 2, 3, 8; 8n+14) - 2N(1, 2, 3, 8; 4n+7).$$

Conjecture 2.5. *Let $n \in \mathbb{N}$ with $n \equiv 0, 2 \pmod{8}$. Then*

$$t(1, 2, 4, 17; n) = 4N(1, 2, 4, 17; n+3).$$

Conjecture 2.6. *Let $n \in \mathbb{N}$. If $n \equiv 2, 3 \pmod{5}$, then*

$$t(1, 1, 5, 8; n) = \frac{1}{2}N(1, 1, 5, 8; 8n+15).$$

Conjecture 2.7. *Let $n \in \mathbb{N}$. If $n \equiv 0, 3, 4, 6, 7 \pmod{9}$, then*

$$t(1, 1, 8, 9; n) = \frac{1}{2}N(1, 1, 8, 9; 8n+19).$$

Conjecture 2.8. *Let $n \in \mathbb{N}$. If $n \equiv 0, 4, 7, 8, 9, 10 \pmod{13}$, then*

$$t(1, 1, 8, 13; n) = \frac{1}{2}N(1, 1, 8, 13; 8n+23).$$

Conjecture 2.9. *Let $n \in \mathbb{N}$. If $n \equiv 0, 3, 5, 6, 7 \pmod{11}$, then*

$$t(1, 1, 4, 11; n) = \frac{1}{3}N(1, 1, 4, 11; 8n+17).$$

Conjecture 2.10. *Let $n \in \mathbb{N}$. If $n \equiv 0, 1, 2, 4, 7 \pmod{11}$, then*

$$t(1, 1, 2, 22; n) = \frac{1}{3}N(1, 1, 2, 22; 8n+26).$$

Conjecture 2.11. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{3}$. Then*

$$t(1, 3, 12, 36; n) = \frac{1}{2}N(1, 3, 12, 36; 8n + 52) - 2N(1, 3, 12, 36; 2n + 13).$$

Conjecture 2.12. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{4}$. Then*

$$t(3, 5, 20, 32; n) = \frac{1}{2}N(3, 5, 20, 32; 8n + 60) - 2N(3, 5, 20, 32; 2n + 15).$$

Conjecture 2.13. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{4}$. Then*

$$t(1, 6, 15, 18; n) = \frac{2}{3}N(1, 6, 15, 18; 8n + 40) - 2N(1, 6, 15, 18; 2n + 10).$$

Conjecture 2.14. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{3}$. Then*

$$t(1, 6, 18, 27; n) = \frac{2}{3}N(1, 6, 18, 27; 8n + 52) - 2N(1, 6, 18, 27; 2n + 13).$$

Conjecture 2.15. *Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{3}$. Then*

$$t(1, 8, 9, 18; n) = \frac{2}{3}N(1, 8, 9, 18; 8n + 36) - 2N(1, 8, 9, 18; 2n + 9).$$

Conjecture 2.16. *Let $n \in \mathbb{N}$ with $4 \mid n$. Then*

$$t(1, 7, 10, 30; n) = \frac{2}{3}N(1, 7, 10, 30; 8n + 48) - 2N(1, 7, 10, 30; 2n + 12).$$

Conjecture 2.17. *Let $n \in \mathbb{N}$ with $n \equiv 3 \pmod{4}$. Then*

$$t(1, 10, 15, 30; n) = \frac{2}{3}N(1, 10, 15, 30; 8n + 56) - 2N(1, 10, 15, 30; 2n + 14).$$

Conjecture 2.18. *Let $n \in \mathbb{N}$ with $n \equiv 2 \pmod{8}$. Then*

$$t(1, 7, 28, 28; n) = \frac{2}{3}N(1, 7, 28, 28; 8n + 64) - 2N(1, 7, 28, 28; 2n + 16).$$

Conjecture 2.19. *Let $n \in \mathbb{N}$ with $n \equiv 8 \pmod{9}$. Then*

$$t(1, 9, 16, 18; n) = \frac{2}{3}N(1, 9, 16, 18; 8n + 44) - 2N(1, 9, 16, 18; 2n + 11).$$

Conjecture 2.20. *Let $n \in \mathbb{N}$ with $n \equiv 1, 7 \pmod{9}$. Then*

$$t(1, 9, 18, 24; n) = \frac{2}{3}N(1, 9, 18, 24; 8n + 52) - 2N(1, 9, 18, 24; 2n + 13).$$

Conjecture 2.21. *Let $n \in \mathbb{N}$ with $n \equiv 1, 4 \pmod{9}$. Then*

$$t(1, 9, 18, 32; n) = \frac{2}{3}N(1, 9, 18, 32; 8n + 60) - 2N(1, 9, 18, 32; 2n + 15).$$

Conjecture 2.22. *Let $n \in \mathbb{N}$ with $n \equiv 5 \pmod{9}$. Then*

$$t(1, 9, 18, 40; n) = \frac{2}{3}N(1, 9, 18, 40; 8n + 68) - 2N(1, 9, 18, 40; 2n + 17).$$

Conjecture 2.23. *Let $n \in \mathbb{N}$ with $n \equiv 2, 5 \pmod{9}$. Then*

$$t(1, 10, 27, 30; n) = \frac{2}{3}N(1, 10, 27, 30; 8n + 68) - 2N(1, 10, 27, 30; 2n + 17).$$

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